On the Construction and Decoding of Cyclic LDPC Codes

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Cyclic LDPC codes form an important class of structured LDPC codes.

- As cyclic codes, they can be simply encoded with shift register.
- As LDPC codes, they provide good performance under iterative decoding with a reasonable decoding complexity.
- They have relatively large minimum distance.
- Some codes can be transformed into quasi-cyclic (QC) codes through row and column permutations on the parity-check matrix.

Some Known Constructions

- Construction based on finite geometries (Lin et al.)
- Construction based on idempotents (Shibuya et al., Tomlinson et al.)
- Construction based on matrix decomposition (Lin et al.)

Code Features

- The defining parity-check matrix is a circulant matrix or a column of circulant matrices.
- The party-check matrix is highly redundant.
- The column weight γ is relatively large.
- The corresponding Tanner graph is free of length-4 cycles.
- The minimum distance of the code is at least $\gamma + 1$. (Massey bound)

Idempotent

Definition: Let R_n be the ring of residue classes of $F_q[x]$ modulo $x^n - 1$. Then a polynomial $e(x)$ of R_n is called an *idempotent* if $e^2(x) = e(x)$.

Properties:

- For a cyclic code C, there exists a unique idempotent $e(x)$ that generates C, called the generating idempotent of C.
- Let $e^{\perp}(x)$ be the generating idempotent of $C^{\perp}$, the dual code of C, then $e^{\perp}(x) = 1 - x^n e(x^{-1})$.

Modular Golomb Ruler

Definition: A set of integers $\{a_i : 0 \leq a_i < n, 1 \leq i \leq \gamma\}$ is called a *Golomb ruler modulo n with* γ *marks*, if the differences $(a_i - a_j)$ mod n are distinct for all ordered pairs (i, j) with $i \neq j$.

An example:

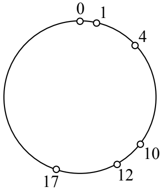


Figure: A Golomb ruler modulo 31 with 6 marks.

An inherent constraint: $\gamma \times (\gamma - 1) \leq (n - 1)$.

Three Algebraic Constructions of Modular Golomb Rulers

- Singer construction (projective geometry plane)
- Bose construction (Euclidean geometry plane)
- Ruzza construction

Code Definition

Consider a cyclic LDPC code C of length n over F_q, whose parity-check matrix is an $n \times n$ circulant

$$\mathbf{H} = \begin{bmatrix} c_0 & c_1 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \cdots & c_0 \end{bmatrix}.$$

The above code is specified by the polynomial

$$\begin{aligned} c(x) &= c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} \\ &= c_{a_1} x^{a_1} + c_{a_2} x^{a_2} \cdots + c_{a_\gamma} x^{a_\gamma}, \end{aligned}$$

where γ is the row weight of $\mathbf{H}$ and $c_{a_i} \neq 0$ $(i = 1, \cdots, \gamma)$.

Main Results

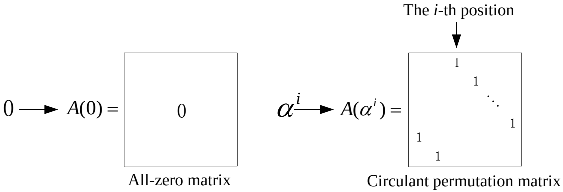
- If $c(x)$ is an idempotent, then the minimum distance of C satisfies

$$d_{min} \leq \begin{cases} \gamma+1, & \text{if } c_0 = 0; \\ \gamma-1, & \text{if } c_0 = 1; \\ \gamma, & \text{otherwise.} \end{cases}$$

- The Tanner graph corresponding to $\mathbf{H}$ is free of length-4 cycles if and only if $\{a_1, a_2, \cdots, a_\gamma\}$ is a Golomb ruler modulo n with γ marks.
- According to Massey bound, if the Tanner graph is free of length-4 cycles, then $d_{min} \geq \gamma + 1$.
- If $c(x)$ is an idempotent and $\{a_1, a_2, \cdots, a_\gamma\}$ is a modular Golomb ruler, then the minimum distance of C is exactly $\gamma + 1$.

Preliminaries: Finite Field based Construction of QC-LDPC Codes (Lin et al.)

Let α be a primitive element of F_q, then $0, \alpha^0, \cdots, \alpha^{q-2}$ give all elements of F_q. Let $\beta \in F_q$, then it can be mapped to a $(q-1) \times (q-1)$ binary matrix $A(\beta)$, as shown below.



The matrix $A(\beta)$ is called the *matrix dispersion* of β over F_2.

Procedure for Finite Field based Construction

- First construct an $m \times n$ matrix over F_q, called the *base matrix*.

$$\mathbf{W} = \begin{bmatrix} \mathbf{w}_0 \\ \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_{m-1} \end{bmatrix} = \begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,n-1} \\ w_{1,0} & w_{1,1} & \cdots & w_{1,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m-1,0} & w_{m-1,1} & \cdots & w_{m-1,n-1} \end{bmatrix}.$$

- Then replace each entry of $\mathbf{W}$ by its matrix dispersion and form the following matrix as the parity-check matrix

$$\mathbf{H}(\mathbf{W}) = \begin{bmatrix} A(w_{0,0}) & A(w_{0,1}) & \cdots & A(w_{0,n-1}) \\ A(w_{1,0}) & A(w_{1,1}) & \cdots & A(w_{1,n-1}) \\ \vdots & \vdots & \ddots & \vdots \\ A(w_{m-1,0}) & A(w_{m-1,1}) & \cdots & A(w_{m-1,n-1}) \end{bmatrix}.$$

Design Constraint on $\mathbf{W}$

For $0 \leq i, j \leq m-1, i \neq j$ and $0 \leq h, k \leq q-2$,

$$d(\alpha^h \mathbf{w}_i, \alpha^k \mathbf{w}_j) \geq n-1,$$

where d denotes the Hamming distance.

The constraint is called the α*-multiplied row distance (RD) constraint*, which guarantees that the Tanner graph corresponding to $\mathbf{H}(\mathbf{W})$ is free of length-4 cycles.

Pseudo-Cyclic Code and MDS Code

Definition: A linear block code of length n is a *pseudo-cyclic* code with parameter $\beta \in F_q$, if for any codeword $(c_0, c_1, \cdots, c_{n-1})$, its pseudo-cyclic $(\beta c_{n-1}, c_0, \cdots, c_{n-2})$ also forms a codeword.

Definition: An (n, k) linear block code is a *maximum-distance-separable* (MDS) code, if the minimum distance $d_{min} = n - k + 1$.

Code Construction

Consider the the following $n \times n$ matrix over F_q

$$\mathbf{W} = \begin{bmatrix} w_0 & w_1 & \cdots & w_{n-2} & w_{n-1} \\ \alpha w_{n-1} & w_0 & \cdots & w_{n-3} & w_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha w_2 & \alpha w_3 & \cdots & w_0 & w_1 \\ \alpha w_1 & \alpha w_2 & \cdots & \alpha w_{n-1} & w_0 \end{bmatrix},$$

where the rows are codewords of a $(n, 2)$ pseudo-cyclic MDS code with $\beta = \alpha$.

It can be proved that the $\mathbf{W}$ satisfies the α-multiplied RD constraint.

Through matrix dispersion, the obtained $\mathbf{H}(\mathbf{W})$ defines a QC-LDPC code.

Main Result

From the above QC-LDPC code, a cyclic LDPC code can be obtained by transforming $\mathbf{H}(\mathbf{W})$ to a circulant parity-check matrix through row and column permutations.

Automorphism Group of a Code

Definition: Let C be a binary linear block code of length n. The set of coordinate permutations that map C to itself forms a group under composition operation. The group is called the *automorphism group* of C, denoted by $Aut(C)$.

For a binary cyclic code of odd length n, the automorphism group contains the following two cyclic subgroups:

- S_0: The set of permutations $\tau^0, \tau^1, \cdots, \tau^{n-1}$, where

$$\tau^k : j \rightarrow (j + k) \, mod \, n.$$

- S_1: The set of permutations $\zeta^0, \zeta^1, \cdots, \zeta^{m-1}$, where

$$\zeta^k : j \rightarrow (2^k \cdot j) \, mod \, n,$$

  and m is the smallest positive integer such that $2^m \equiv 1 \, mod \, n$.

Properties of $Aut(C)$

- Let $C^{\perp}$ be the dual code of C, then $Aut(C^{\perp}) = Aut(C)$.
- Let $\pi \in Aut(C)$. If $\mathbf{H}$ is a parity-check matrix of C, then $\pi\mathbf{H}$ also forms a parity-check matrix of C.

Decoder Diversity

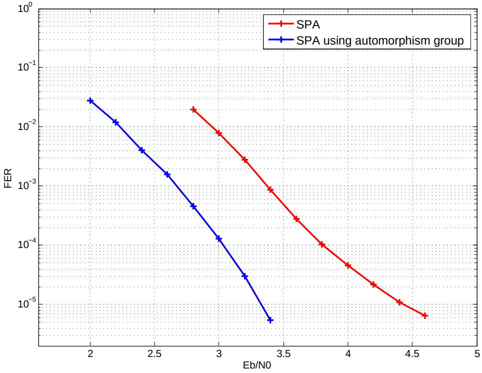
- Two party-check matrices are called *non-equivalent* if they cannot be obtained from each other only by row permutations.
- The basic idea is to construct multiple non-equivalent parity-check matrices based on $Aut(C)$. Different decoding attempts can be made on these parity-check matrices, thus providing decoder diversity gain.

Main Results

- For cyclic LDPC codes constructed from idempotents and modular Golomb rulers, S_0 and S_1 cannot be used to generate non-equivalent parity-check matrices.
- For cyclic LDPC codes constructed from pseudo-cyclic MDS codes with two information symbols, S_1 can be used to generate non-equivalent parity-check matrices.

Simulation Results

A $(341, 160)$ cyclic LDPC code is constructed from $(31, 2)$ pseudo-cyclic MDS code over F_{32}. The BPSK modulation over AWGN channel is assumed. The maximum number of iterations is set to be 100.



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Thanks!